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PURE TWIST OF A SOLID CIRCULAR RING SECTOR

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Abstract—The pure twist of an incomplete toroidal ring sector of solid circular cross-section has been solved by previous investigators. If we impose the requirement that a solution is to be determined such that it has the capability of being generalized to non-isotropic circular cross-sections and to hollow circular cross-sections, the only methods available appear to be those of "toroidal elasticity".

The previous solution by Göhner can not be extended to hollow circular cross-sections (toroidal tubes). The previous solution by Freiberger uses toroidal coordinates which can not be applied readily to non-isotropic cross-sections nor to circular cross-sections of uniform wall thickness. To overcome these limitations in the two previous solutions, it is necessary to introduce the coordinate frame of toroidal elasticity as illustrated in the present paper. The extensions to non-isotropy and to hollow cross-sections will be demonstrated in two separate follow-on papers.

NOTATION

ρ, η, θ	Göhner's coordinate system
r, ϕ, θ	T.E. coordinate system
$\tau_{r\theta}, \tau_{\phi\theta}$	shear stresses ($\sigma_r = \sigma_\phi = \sigma_\theta = \tau_{r\phi} = 0$) T.E. (abbreviation for toroidal elasticity)
w	T.E. displacement ($u = v = 0$)
R	toroidal radius
s	r/R
ρ	$R + r\cos\phi$
q	$1 + s\cos\phi = \rho/R$
a	radius of circular cross-section
c	constant
Φ	stress function
G	shear modulus
M_t	PR twisting moment

INTRODUCTION

The twist of a ring sector is of practical interest in calculating stress fields in close-coiled helical springs. The problem has been solved by several investigators[1–5]. A history of previous solutions was discussed in the paper of Freiberger[5]. He uses toroidal coordinates in his solution. Earlier, Göhner[1–3] also derived a solution using the method of successive approximation. A compact account of his analysis appears in Timoshenko and Goodier[4].

Both solutions exhibit serious limitations. They cannot be readily extended to hollow ring sectors of uniform cross-sections nor can they be readily extended to non-isotropic materials. Both of these limitations can be overcome by introducing an appropriate new frame of reference and generating displacements and stresses. This frame of reference leads to a fully three-dimensional theory of elasticity which will be referred to as "toroidal elasticity" (or TE, for brevity). Some of the methods of TE are generated in this paper.

The advantages claimed for TE are:

(1) The introduction of a method capable of extension to ring sectors of hollow circular cross-sections (to be discussed in a separate paper).

(2) The introduction of a method capable of extension to non-isotropic ring sectors (to be discussed in a separate paper).

(3) Simplification in algebraic details when compared to methods based on toroidal coordinates.

(4) A verification of the new compatibility equations which are required. It is to be noted that numerical results for all three approaches (i.e. Göhner, Freiberger and Lang) should yield the same values for stresses.

HARMONIC OPERATOR

The first step in the analysis is to obtain the harmonic operator ∇_0^2 in the (s, ϕ) system. From Fig. 1, the equations connecting the two coordinate systems are

$$\eta = r \sin \phi \tag{1}$$

$$\rho = R + r \cos \phi = qR. \tag{2}$$

We introduce the operators

$$L_q = R \frac{\partial}{\partial \rho} = \cos \phi \frac{\partial}{\partial s} - \frac{\sin \phi}{s} \frac{\partial}{\partial \phi}$$

$$L_\eta = R \frac{\partial}{\partial \eta} = \sin \phi \frac{\partial}{\partial s} + \frac{\cos \phi}{s} \frac{\partial}{\partial \phi}$$

where $s = r/R$. Then the operator of Göhner transforms

$$R^2\left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \eta^2}\right) \text{ to } L_q^2 + L_\eta^2 = \nabla_0^2$$

where

$$\nabla_0^2 = \frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s^2}\frac{\partial^2}{\partial \phi^2}. \tag{3}$$

The last expression on the right is required for the T.E. coordinate system.

THE EQUILIBRIUM EQUATION

The stresses σ_r, σ_ϕ, σ_θ and $\tau_{r\phi}$ vanish. The displacements u and v also vanish. A single equilibrium equation remains and assumes the form

$$\frac{\partial}{\partial s} \tau_{r\theta} + \frac{1}{s} \tau_{r\theta} + \frac{1}{s} \frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{2}{q} (\tau_{r\theta} \cos \phi - \tau_{\phi\theta} \sin \phi) = 0. \tag{4}$$

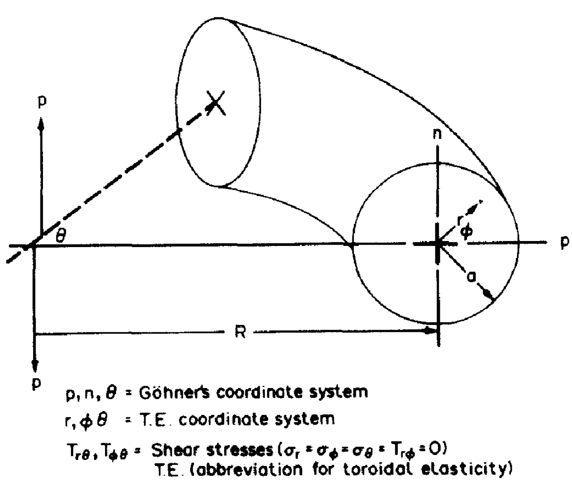

Fig. 1. Ring sector under pure twist.

This equation is identically satisfied by introducing the stress function, ϕ such that

$$\tau_{r\theta} = \frac{1}{q^2 s}\frac{\partial \Phi}{\partial \phi} \tag{5}$$

$$\tau_{\phi\theta} = -\frac{1}{q^2}\frac{\partial \Phi}{\partial s}. \tag{6}$$

In addition, it is noted that the boundary condition is particularly simple and reduces to $Y_{r\theta} = 0$ at $r = a$(or $s = s_a$).

COMPATIBLE EQUATIONS

The compatibility equations for the shear stresses $\tau_{r\theta}$ and $\tau_{\phi\theta}$ are:

$$\nabla_1^2\tau_{r\theta} - \frac{\tau_{r\theta}}{s^2} - \frac{2}{s^2}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{\tau_{\phi\theta}\sin\phi}{qs}$$

$$-\frac{1}{q^2}[\tau_{r\theta} + 3\cos\phi(\cos\phi\tau_{r\theta} - \sin\phi\tau_{\phi\theta})] = 0 \tag{7}$$

$$\nabla_1^2\tau_{\phi\theta} - \frac{\tau_{\phi\theta}}{s^2} + \frac{2}{s^2}\frac{\partial \tau_{r\theta}}{\partial \phi} - \frac{\tau_{r\theta}\sin\phi}{qs}$$

$$-\frac{1}{q^2}[\tau_{\phi\theta} + 3\sin\phi(\sin\phi\tau_{\phi\theta} - \cos\phi\tau_{r\theta})] = 0. \tag{8}$$

Equations (7) and (8) were obtained by transforming the isotropic stress compatibility equations (see [4], p. 232) from a rectangular coordinate frame to the reference frame (s, ϕ, θ). Here $\nabla_1^2 = \nabla_0^2 + (1/q)L_q$.

Guided by Göhner's solution, we confirm that the expression

$$\nabla_1^2\left(\frac{Y}{q^2}\right) = \frac{1}{q^2}\left(\nabla_0^2 - \frac{3}{q}L_q\right)Y + \frac{4Y}{q^2}$$

is an identity. We apply it first to $Y_1 = (1/s)(\partial\Phi/\partial\phi)$ and then to $Y_2 = -(\partial\Phi/\partial s)$. The first compatibility equation reduces to

$$\frac{\partial}{s\partial\phi}\left[\nabla_0^2\Phi - \frac{3}{q}L_q\Phi\right] = 0 \tag{9}$$

the second compatibility equation reduces to

$$\frac{\partial}{\partial s}\left[\nabla_0^2\Phi - \frac{3}{q}L_q\Phi\right] = 0. \tag{10}$$

In effecting the transformations which lead to eqns (9) and (10), additional terms appear. These terms are easily grouped in such a manner that they are identically zero.

The expression in the brackets is, therefore, a constant on the boundary of the circular cross-section. We take the constant to be $-2cR^2$ so the equation governing the stress function, Φ is:

$$\nabla_0^2\Phi - \frac{3}{q}L_q\phi + 2cR^2 = 0 \tag{11}$$

in complete agreement with Göhner.

The stress function is determined by writing the series

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 + \cdots$$

and applying the method of successive approximations, where

$$\nabla_0^2\Phi_0 + 2cR^2 = 0$$

$$\nabla_0^1\Phi_1 - 3\frac{\partial\Phi_0}{\partial q} = 0$$

$$\nabla_0^2\Phi_2 - 3\frac{\partial\Phi_1}{\partial q} + 3s\cos\phi\frac{\partial\Phi_0}{\partial q} = 0$$

$$\nabla_0^2\Phi_3 - 3\frac{\partial\Phi_2}{\partial q} + 3s\cos\phi\frac{\partial\Phi_1}{\partial q} - 3s^2\cos^2\phi\frac{\partial\Phi_0}{\partial q} = 0. \quad (12)$$

For the purposes of this paper, it is sufficient to restrict the series to four terms, Φ_0 to Φ_3. Corresponding expressions can be developed for the two shear stresses. The shear stresses are determined by the expressions

$$(\tau_{r\theta})_0 = G/Rs\cdot\frac{\partial\Phi_0}{\partial\phi} \quad (\tau_{r\theta})_1 = (\tau_{r\theta}) + G/Rs\cdot\frac{\partial\Phi_1}{\partial\phi}$$

$$(\tau_{r\theta})_2 = (\tau_{r\theta})_1 + \frac{G}{R}\left(\frac{1}{s}\frac{\partial\Phi_2}{\partial\phi} - 2\cos\phi\frac{\partial\Phi_1}{\partial\phi}\right)$$

$$(\tau_{r\theta}) = (\tau_{r\theta})_2 + \frac{G}{R}\left(\frac{1}{s}\frac{\partial\Phi_3}{\partial\phi} - 2\cos\phi\frac{\partial\Phi_2}{\partial\phi} + 3s\cos^2\phi\frac{\partial\phi_1}{\partial\phi}\right) \quad (13)$$

and by

$$(\tau_{\phi\theta})_0 = -\frac{G}{R}\cdot\frac{\partial\Phi_0}{\partial s}$$

$$(\tau_{\phi\theta})_1 = (\tau_{\phi\theta})_0 - \frac{G}{R}\left(\frac{\partial\Phi_1}{\partial s} - 2s\cos\phi\frac{\partial\Phi_0}{\partial s}\right)$$

$$(\tau_{\phi\theta})_2 = (\tau_{\phi\theta})_1 - \frac{G}{R}\left(\frac{\partial\Phi_2}{\partial s} - 2s\cos\phi\frac{\partial\Phi_1}{\partial s} + 3s^2\cos\phi\frac{\partial\Phi_0}{\partial s}\right)$$

$$(\tau_{\phi\theta})_3 = (\tau_{\phi\theta})_2 - \frac{G}{R}\left(\frac{\partial\Phi_3}{\partial s} - 2s\cos\phi\frac{\partial\Phi_2}{\partial s} + 3s^2\cos^2\phi\frac{\partial\Phi_1}{\partial s} - 4s^3\cos^3\phi\frac{\partial\Phi_0}{\partial s}\right) \quad (14)$$

THE STRESS FIELD (0)

The first stress function equation $\nabla_0^2\Phi_o + 2cR^2 = 0$ has the solution $\Phi_0 = -(cR^2s^2/2)$. The stresses corresponding to Φ_0 are $(\tau_{\phi\theta})_0 = GcRs$ $(\tau_{r\theta})_0 = 0$. Corresponding to the torsion of a straight rod of circular cross-section.

THE STRESS FIELD (1)

Since $(\partial\Phi_0/\partial q) = -cR^2s\cos\phi$ we have

$$\nabla_0^2\Phi_1 = -3cR^2s\cos\phi.$$

The boundary condition is satisfied by adding the harmonic solution $(3/8)cs_a^2R^2(s\cos\phi)$ to the particular solution $-(3/8)cR^3\cos\phi$ to obtain

$$\Phi_1 = \frac{3}{8}cR^2s(s_a^2 - s^2)\cos\phi.$$

The stresses corresponding to Φ_1 are

$$(\tau_{r\theta})_1 = \frac{3}{8}GcR(s^2 - s_a^2)\sin\phi(\Upsilon_{\phi\theta})_1 = G_cR\left[s - \left(\frac{3}{8}s_a^2 + \frac{7}{8}s^2\right)\cos\phi\right].$$

THE STRESS FIELD (2)

The next equation is: $\nabla^2_{02} = (3/4)cR^2s^2\cos\phi + (9/8)cR^2(s_a^2 - s^2)$. To the particular solution, we add the harmonic term

$$\frac{-s_a^2cR^2s^2\cos 2\phi}{32}.$$

The final result is

$$\Phi_2 = cR^2\left[\frac{5}{16}s_a^2s^2 - \frac{5s^4}{64} - \frac{s^2\cos^2\phi}{16}(s_a^2 - s^2)\right]$$

generating the stresses

$$\begin{aligned}(\tau_{\phi\theta})_2 &= GcR\left[s - \left(\frac{3}{8}s_a^2 + \frac{7}{8}s^2\right)\cos\phi\right]\\ &\quad + GcRs\left[\frac{5}{16}s^2 - \frac{5}{8}s_a^2\right]\\ &\quad + GcRs\cos^2\phi\left[\frac{s^2}{2} + \frac{7}{8}s_a^2\right]\\ (\tau_{r\theta})_2 &= \frac{3}{8}GcR(s^2 - s_a^2)\sin\phi\\ &\quad + \frac{7}{8}GcR(s_a^2 - s^2)s^2\cos\phi\sin\phi.\end{aligned}$$

THE STRESS FIELD (3)

Since the method should be apparent, the next results are merely listed.

$$\begin{aligned}\Phi_3 &= cR^2s\cos\phi\left(\frac{15}{512}s^4 + \frac{15}{512}s_a^2s^2 - \frac{15}{256}s_a^4\right)\\ &\quad + cR^2\left(\frac{3}{128}s^3\right)\cos^3\phi(s_a^2 - s^2)\end{aligned}$$

$$\begin{aligned}(\tau_{r\theta})_3 &= \frac{3}{8}GcR(s^2 - s_a^2)\sin\phi - \frac{7}{8}GcR\cos\phi\sin\phi s(s^2 - s_a^2)\\ &\quad + \frac{15}{512}GcR\sin\phi(2s_a^4 - s_a^2s^2 - s^4)\\ &\quad + \frac{185}{128}GcR\cos^2\phi\sin\phi(s^2 - s_a^2)s^2\end{aligned}$$

$$\begin{aligned}(\tau_{\phi\theta})_3 &= GcRs\left(1 + \frac{5}{16}s^2 - \frac{5}{8}s_a^2\right)\\ &\quad + GcR\cos\phi\left(-\frac{3}{8}s_a^2 - \frac{7}{8}s^2 + \frac{595s_a^2s^2}{512} - \frac{395s^4}{512}\right)\\ &\quad + GcR\cos^2\phi s\left(\frac{s^2}{2} + \frac{7}{8}s_a^2\right)\\ &\quad - GcR\cos^3\phi s^2\left(\frac{185}{128}s_a^2 + \frac{1}{128}s^2\right).\end{aligned}$$

DETERMINATION OF THE CONSTANT, c

The constant, c, is determined by the condition that the shear stress on the cross-section equates the resultant twisting moment $M_t = PR$. This condition is

$$M_t = PR = \int_0^{2\pi} \int_0^a (\Upsilon_{\phi\theta} r) r \, \mathrm{d}r \, \mathrm{d}\phi = 2\pi R^3 \int_0^{s_a} (\Upsilon_{\phi\theta}) s^2 \, \mathrm{d}s.$$

For solutions (0) and (1), the result is

$$M_t = PR = Gc \frac{\pi a^4}{2} = GcI_p$$

so

$$Gc = \frac{2PR}{\pi a^4}. \tag{15}$$

For solutions (2) and (3), the result is

$$M_t = PR = GcI_p \left(1 + \frac{3}{16} s_a^2\right)$$

so

$$Gc = \frac{2P}{\pi a^3} \frac{1}{s_a \left(1 + \frac{3}{16} s_a^2\right)}. \tag{16}$$

Both results agree with Göhner's determination of c.

VERIFICATION OF AGREEMENT WITH GÖHNER'S RESULTS

Göhner determined

$$(\tau_{\rho\theta})_1 = -cG \left(\eta + \frac{5}{8} \eta \left(\frac{R-\rho}{R}\right)\right)_2$$

$$(\tau_{\eta\theta})_1 = -cG \left(R - \rho + \frac{7}{8} \frac{(R-\rho)^2}{R} - \frac{3}{8} \left(\frac{\eta^2 - a^2}{R}\right)\right).$$

If we form the same stresses from the equations

$$(\tau_{\rho\theta})_1 = (\tau_{r\theta})_1 \cos\phi - (\tau_{\phi\theta})_1 \sin\phi$$

$$(\tau_{\eta\theta})_1 = (\tau_{r\theta})_1 \sin\phi + (\tau_{\phi\theta})_1 \cos\phi.$$

the agreement is exact, as it must be.

THE DISPLACEMENT, w

The strain-displacement equations are

$$Re_{r\theta} = \frac{\partial w}{\partial s} - \frac{w \cos\phi}{q} = q \frac{\partial}{\partial s}\left(\frac{w}{q}\right)$$

$$Re_{\phi\theta} = \frac{1}{s} \frac{\partial w}{\partial \phi} + \frac{w \sin\phi}{q} = \frac{q}{s} \frac{\partial}{\partial \phi}\left(\frac{w}{q}\right). \tag{17}$$

These are difficult to use for the determination of the displacement, w. We revert to Göhner's system to obtain $e_{\eta\theta} = (\partial w/\partial \eta)$ which is easy to integrate. The first three displacement equations are $w_0 = c\eta(\rho - R)$

$$w_1 = w_0 - \frac{5}{4}\frac{c\eta}{R}(\rho - R)^2 + \frac{3}{8}\frac{c}{R}\left(\frac{\eta^3}{3} + \eta(\rho - R)^2 - a^2\eta\right)$$

$$w_2 = w_1 + \frac{1}{4}\frac{a^2c}{R^2}\cdot(\rho - R)\eta + \frac{13c}{16R^2}\eta(\rho - R)^3 - \frac{3}{16}\frac{c}{R^2}\eta^3(\rho - R).$$

These could be rewritten in the (r, ϕ) system using eqns (1) and (2). Since it is always simple to pass from one coordinate frame to the other, the method used to determine w is frequently useful. The displacement, w, now becomes (in the r, ϕ system)

$$w_0 = cR^2(s^2 \sin\phi \cos\phi)$$

$$w_1 = cR^2(s^2 \sin\phi\cos\phi) - \frac{3}{8}cR^2(ss_a^2 \sin\phi) + \frac{cR^2s^3}{8}\sin\phi$$

$$- cR^2s^3 \sin\phi\cos^2\phi, \text{ etc.} \tag{18}$$

Finally, $e_{r\theta}$ and $e_{\phi\theta}$ can be determined by differentiation only using eqn (17).

NUMERICAL RESULTS

The stresses $(\tau_{\phi\theta})_3$ and $(\tau_{r\theta})_3$ are determined for five values of a/R (see Tables 1–3). Along a diameter ($\phi = 0$ or 180°), we have

$$\tau_{\phi\theta} = GcR\left(s + \frac{13}{16}s^3 + \frac{1}{4}ss_a^2\right) \pm GcR\left(\frac{3}{8}s_a^2 + \frac{7}{8}s^2\right)$$

$$\pm GcR\left(\frac{399s^4}{512} + \frac{145s^2s_a^2}{512}\right)$$

(where the lower sign corresponds to $\phi = 0$).

Table 1. Numerical values for $\dfrac{\tau_{\phi\theta}}{\frac{2PR}{\pi a^3}}$ along horizontal diameter

$\left(\dfrac{\tau_{\phi\theta}}{\frac{2PR}{\pi a^3}}\right)$ at $\phi = 180°$

r/a ↓ \ a/R →	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
1.0	1.1354	1.2992	1.3948	1.5773	2.0522
0.75	1.0617	0.9408	0.9946	1.1119	1.3056
0.50	0.5762	0.6361	0.6574	0.7141	0.8348
0.25	0.2931	0.3364	0.3582	0.3941	0.4670
0.0	0.0374	0.0744	0.0927	0.1224	0.1672

$\left(\dfrac{\tau_{\phi\theta}}{\frac{2PR}{\pi a^3}}\right)$ at $\phi = 0°$

r/a ↓ \ a/R →	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
1.0	0.8820	0.7702	0.7132	0.6113	0.3645
0.50	0.4422	0.3745	0.3590	0.3144	0.2286

Table 2. Numerical values for $\frac{\tau_{\phi\theta}}{\frac{2PR}{\pi a^3}}$ along vertical diameter

$\left(\frac{\tau_{\phi\theta}}{\frac{2PR}{\pi a^3}}\right)$ at $\phi = \pm 90°$

r/a ↓ \ a/R →	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
0.25	0.2480	0.2421	0.2377	0.2283	0.2027
0.5	0.4964	0.4854	0.4772	0.4600	0.4123
1.0	0.9951	0.9801	0.9690	0.9448	0.8805

Table 3. Numerical values for $\frac{\tau_{r\theta}}{\frac{2PR}{\pi a^3}}$ along vertical diameter

$\left(\frac{\tau_{r\theta}}{\frac{2PR}{\pi a^3}}\right)$ at $\phi = 90°$

r/a ↓ \ a/R →	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
0	-0.3740	-0.3699	-0.3671	-0.3610	-0.3443
0.25	-0.3509	-0.3489	-0.3472	-0.3440	-0.3349
0.50	-0.2805	-0.2772	-0.27494	-0.2701	-0.2568
0.75	-0.1637	-0.1628	-0.16192	-0.1603	-0.1560
1.0	0	0	0	0	0

($\tau_{r\theta}$ reverses sign when $\phi = -90°$ or $270°$)

Along a vertical diameter ($\phi = 90°$), we have

$$\tau_{r\theta} = GcR\left(s + \frac{5}{16}s^3 - \frac{5}{8}ss_a^2\right).$$

Along a horizontal diameter ($\phi = 0$ or $180°$), the stress $\Upsilon_{r\theta}$ vanishes, but along a vertical diameter

$$\tau_{\phi\theta} = \frac{3}{8}GcR(s^2 - s_a^2) + \frac{15}{512}GcR(2s_a^4 - s_a^2s^2 - s^4).$$

The sign of $\tau_{r\theta}$ reverses for $\phi = -90°$.

CONCLUSIONS

Figures 2–4 show the two shear stresses in dimensionless form, in particular, Fig. 2 indicates the departure of the shear stress $\tau_{\phi\theta}$ from the linear variation associated with a straight twisted solid circular shaft. These figures are drawn for three values of $a/R = 1/2$, $1/3$, $1/5$.

The tables and the figures, collectively, should enable stress engineers to extrapolate the two stresses and the stress displacement, w, to their own particular toroidal geometries.

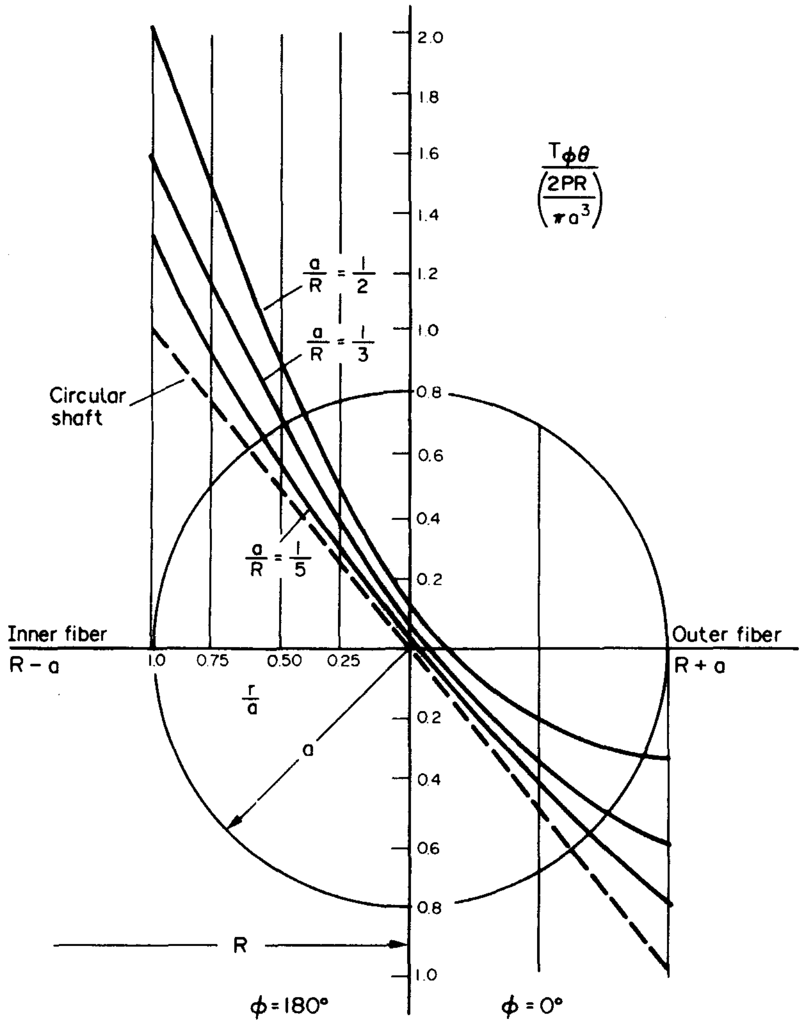

Fig. 2. $\frac{T_{\phi\theta}}{\frac{2PR}{\pi a^3}}$ along horizontal diameter.

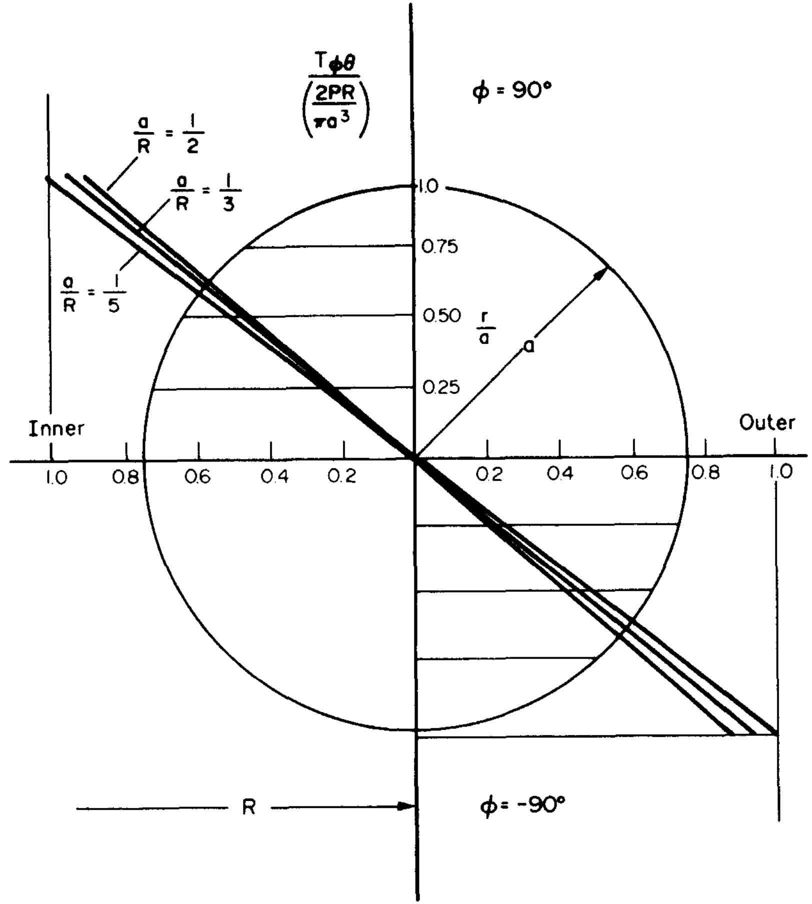

Fig. 3. $\frac{T_{\phi\theta}}{\frac{2PR}{\pi a^3}}$ along vertical diameter.

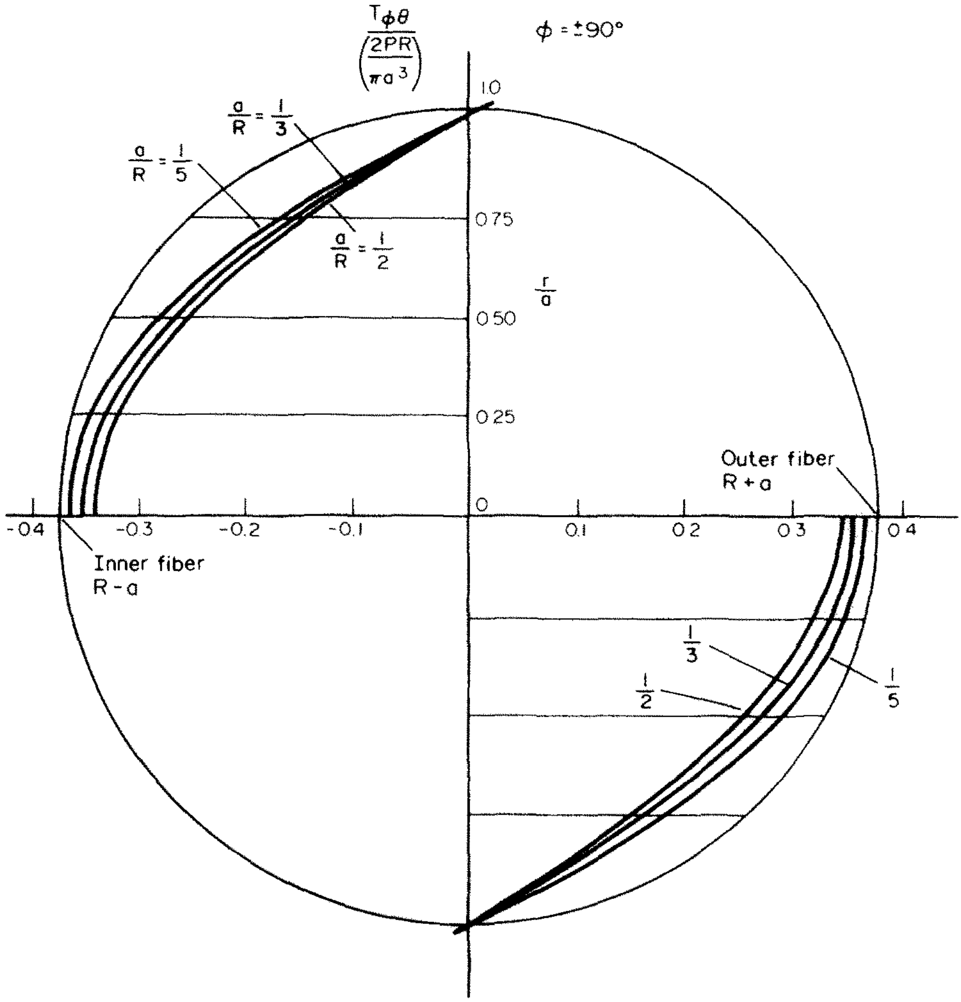

Fig. 4. $\frac{\tau_{r\theta}}{\frac{2PR}{\pi a^3}}$ along vertical diameter.

The methods and equations developed in this paper will be subsequently applied to the related problem of twist of a ring sector where the cross-section is hollow.

REFERENCES

1. O. Göhner, Schubspannungsverteilung im Querschnitt einer Schraubenfeder. *Ingenieur Archiv* **1**, 619–644 (1930).
2. O. Göhner, Schubspannungsverteilung im Querschnitt eines gedrillten Ringstabs mit Anwendung auf Schraubenfedern. *Ingenieur Archiv* **2**, 1–19 (1931).
3. O. Göhner, Spannungsverteilun in einem an den Endquerschnitten belasteten Ringstabsektor. *Ingenieur Archiv* **2**, 381–414 (1931).
4. S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 2nd Edn, pp. 391–398. McGraw-Hill, New York (1951).
5. W. Freiberger, The uniform torsion of an incomplete tore. *Australian J. Scientific Res. Series A*, **2**, 354–375 (1949).